

ON THE DIMENSION THEORY OF OVERRINGS OF AN INTEGRAL DOMAIN⁽¹⁾

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Let D be an integral domain with identity which has quotient field L . If there exists a chain $P \subset P_1 \subset \dots \subset P_n$ of $n+1$ prime ideals of D , where $P_n \subset D$, but no such chain of $n+2$ prime ideals, then we say that D has dimension n and we write $\dim D = n$ [6]. In [6] and [7] Seidenberg has shown that if $\dim D = n$, and if D is a Noetherian domain or a Prüfer domain, then $\dim D[X_1, \dots, X_m] = n+m$, where $X_1, \dots, X_m$ are indeterminates over D . In the special case in which $\dim D = 1$ he has proved that the following statements are equivalent.

- (1) $\dim D[X_1] = 2$.
- (2) $\dim D[X_1, \dots, X_m] = m+1$ for any m .

More recently Gilmer has established the equivalence of the following properties for an n -dimensional domain D [1].

- (3) Every domain between D and L has dimension less than or equal to n .
- (4) $\dim D[t_1, \dots, t_n] \leq n$ for $\{t_1, \dots, t_n\} \subseteq L$.

For $n=1$ he further showed that (3) and (4) are equivalent to (1).

In this paper we consider domains D having finite dimension n and having the property that each domain between D and its quotient field has dimension less than or equal to ω for some positive integer $\omega \geq n$. For such a domain we obtain equivalent statements analogous to statements (1)–(4). The main results of this paper are contained in Theorems 2 and 5.

Throughout this paper D will denote an integral domain with identity having quotient field L , and $X, X_1, \dots, X_m$ will denote indeterminates over D . By an *overring* of D we mean an integral domain D' such that $D \subseteq D' \subseteq L$. By a *valuation overring* of D we mean an overring of D which is a valuation ring. Our notation will be that of Zariski-Samuel [8] with the one exception: $\subset$ denotes proper containment and $\subseteq$ denotes containment.

I. If $\dim D = n$ and $\omega \geq n$, we wish to find necessary and sufficient conditions in order that each overring of D have dimension less than or equal to ω . One such set of conditions is given by the following theorem.

THEOREM 1. *Suppose that $\dim D = n$. Then the following statements are equivalent.*

- (1) *Each overring of D has dimension less than or equal to ω .*

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(2) *Each valuation overring of D has dimension less than or equal to ω .*

(3) *For $\{t_1, \dots, t_\omega\} \subseteq L$, $\dim D[t_1, \dots, t_\omega] \leq \omega$.*

Proof. Clearly (2) and (3) follow from (1). To show that (2) implies (1) we suppose that D' is an overring of D such that $\dim D' > \omega$. Then there exists a chain $(0) \subset P_1 \subset \dots \subset P_s \subset D'$ of prime ideals of D' such that $s > \omega$. By Theorem 11.9 of [5, p. 37] there exists a valuation overring V of D' such that V has prime ideals $Q_1, \dots, Q_s$ which lie over $P_1, \dots, P_s$ respectively. It follows that V is a valuation overring of D such that $\dim V > \omega$.

The proof that (3) implies (1) is given by Gilmer in [1] for the case $\omega = n$. The same method of proof is used here.

Suppose that there exists an overring D' of D such that $\dim D' \geq \omega + 1$. Let $(0) \subset P_{\omega+1} \subset \dots \subset P_1 \subset D'$ be a chain of prime ideals of D' and let $t_i \in P_i - P_{i+1}$, $1 \leq i \leq \omega$. If $P'_i = P_i \cap D[t_1, \dots, t_\omega]$, $1 \leq i \leq \omega + 1$, then P'_i is a prime ideal of $D[t_1, \dots, t_\omega]$ and $t_i \in P'_i - P'_{i+1}$, $1 \leq i \leq \omega$. Now let $r/s \in P_{\omega+1}$, where $r, s \in D - \{0\}$. Then $r = s(r/s)$ is an element of $P_{\omega+1} \cap D$ so that $r \in P'_{\omega+1}$. Further, $1 \notin P'_1$ since $1 \notin P_1$. Thus, $(0) \subset P'_{\omega+1} \subset \dots \subset P'_1 \subset D[t_1, \dots, t_\omega]$ and $\dim D[t_1, \dots, t_\omega] \geq \omega + 1$.

Theorem 1 leads us to the consideration of domains D such that $\dim D[t_1, \dots, t_\omega] \leq \omega$ for any subset $\{t_1, \dots, t_\omega\}$ of L . More generally, for a fixed positive integer m , we wish to find necessary and sufficient conditions in order that $\dim D[t_1, \dots, t_m] \leq \omega$, where $\{t_1, \dots, t_m\}$ is any subset of L , and where $\omega \geq \dim D$. Sufficient conditions are given by the following theorem.

THEOREM 2. *Suppose that $\dim D[X_1, \dots, X_m] = \omega + m$. Then $\omega \geq \dim D$, and given $\{t_1, \dots, t_m\} \subseteq L$, we have $\dim D[t_1, \dots, t_m] \leq \omega$.*

If P is a prime ideal of an integral domain R , we shall denote by $h(P)$ ($d(P)$) the height (depth) of P in R . Before proving Theorem 2, we require Lemma 1.

LEMMA 1. *Let $t_1, \dots, t_m$ be elements of L and let ϕ be the canonical D -homomorphism from $D[X_1, \dots, X_m]$ onto $D[t_1, \dots, t_m]$ such that $\phi(X_i) = t_i$, $1 \leq i \leq m$. If Q is the kernel of ϕ , then Q has height m in $D[X_1, \dots, X_m]$.*

Proof. We have $Q \cap D = (0)$, for if $d \in D$, then $\phi(d) = d$. Thus, if $N = D - \{0\}$, Q extends to a proper prime ideal of

$$(D[X_1, \dots, X_m])_N = D_N[X_1, \dots, X_m] = L[X_1, \dots, X_m].$$

Further, if $h(Q) = s$ in $D[X_1, \dots, X_m]$, the extension of Q has height s in

$$L[X_1, \dots, X_m].$$

However, $\dim L[X_1, \dots, X_m] = m$ so that $s \leq m$.

Let $Q_i = Q \cap D[X_1, \dots, X_i]$ for $1 \leq i \leq m$. If $t_i = a_i/b_i$, $a_i, b_i \in D$, then $b_i X_i - a_i \in Q_i$. Consequently, $Q_i \neq (0)$ and for $1 \leq i \leq m-1$, $Q_i[X_{i+1}] \subset Q_{i+1}$, since $b_{i+1} X_{i+1} - a_{i+1} \notin Q_i[X_{i+1}]$. It now follows that $(0) \subset Q_1[X_2, \dots, X_m] \subset \dots \subset Q_{m-1}[X_m] \subset Q_m = Q$ so that $h(Q) \geq m$. Thus equality holds and the lemma is proved.

Proof of Theorem 2. Suppose that $\{t_1, \dots, t_m\} \subseteq L$ and let Q be the kernel of the D -homomorphism ϕ of $D[X_1, \dots, X_m]$ onto $D[t_1, \dots, t_m]$, where ϕ is such that $\phi(X_i) = t_i$ for each i . Further, let $(0) \subset Q_1 \subset \dots \subset Q_k \subset D[t_1, \dots, t_m]$ be a chain of prime ideals of $D[t_1, \dots, t_m]$. Then there exists a chain

$$Q \subset P_1 \subset \dots \subset P_k \subset D[X_1, \dots, X_m]$$

of prime ideals of $D[X_1, \dots, X_m]$ such that $\phi(P_i) = Q_i$, $1 \leq i \leq k$. Since Q has height m , $m + k \leq \dim D[X_1, \dots, X_m] = m + \omega$. Therefore, $k \leq \omega$ as we wished to show.

Theorem 5 shows that the conditions given in Theorem 2 are also necessary in order that $\dim D[t_1, \dots, t_m] \leq \omega$ for $\{t_1, \dots, t_m\} \subseteq L$. However, before proving Theorem 5 we need several other results.

THEOREM 3. Suppose that $\dim D[t_1, \dots, t_m] \leq \omega$ for $\{t_1, \dots, t_m\} \subseteq L$. If J is an integral domain containing D such that J is integral over D , and if F is the quotient field of J , then $\dim J[s_1, \dots, s_m] \leq \omega$ for $\{s_1, \dots, s_m\} \subseteq F$.

In order to prove Theorem 3, we use the following lemma.

LEMMA 2. Let $f(X) = f_n X^n + \dots + f_1 X + f_0 \in D[X]$, $f_n \neq 0$, and let s be a root of $f(X)$ in an extension field of L . Then s is integral over $D[1/f_n]$ and $f_n s$ is integral over D .

Proof. Since $f(X)/f_n \in D[1/f_n][X]$ and $f(s)/f_n = 0$, it follows that s is integral over $D[1/f_n]$. Also

$$0 = f_n^{n-1} f(s) = (f_n s)^n + f_{n-1} (f_n s)^{n-1} + \dots + f_1 f_n^{n-2} (f_n s) + f_0 f_n^{n-1}$$

so that $f_n s$ is integral over D .

Proof of Theorem 3. F is algebraic over L since J is integral over D . Therefore, if $s_1, \dots, s_m$ are elements of F , there exists $f_i(X) \in D[X] - \{0\}$ such that $f_i(s_i) = 0$, $1 \leq i \leq m$. It follows from Lemma 2 that if d_i is the leading coefficient of $f_i(X)$, then s_i is integral over $D[1/d_1, \dots, 1/d_m]$ for each i , $1 \leq i \leq m$. Hence, $J[s_1, \dots, s_m]$ is integral over $D[1/d_1, \dots, 1/d_m]$. Therefore, $\dim J[s_1, \dots, s_m] = \dim D[1/d_1, \dots, 1/d_m]$ by [6, Theorem 5]. But $\dim D[1/d_1, \dots, 1/d_m] \leq \omega$ since $\{1/d_1, \dots, 1/d_m\} \subseteq L$.

COROLLARY 1. Suppose that $\dim D[t_1, \dots, t_m] \leq \omega$ for $\{t_1, \dots, t_m\} \subseteq L$. Then if $\{s_1, \dots, s_m\}$ is a set of elements algebraic over D , we have $\dim D[s_1, \dots, s_m] \leq \omega$.

Proof. Suppose $f_i(X) \in D[X] - \{0\}$ is such that $f_i(s_i) = 0$, $1 \leq i \leq m$. From Lemma 2 it follows that if d_i is the leading coefficient of $f_i(X)$, then $J = D[d_1 s_1, \dots, d_m s_m]$ is integral over D . Moreover, $\{s_1, \dots, s_m\}$ is a subset of the quotient field of J . Therefore, since $J[s_1, \dots, s_m] = D[s_1, \dots, s_m]$, it follows from Theorem 3 that $\dim D[s_1, \dots, s_m] \leq \omega$.

THEOREM 4. If each overring of D has dimension less than or equal to ω , then $\dim D[X_1, \dots, X_m] \leq \omega + m$.

Proof. Suppose that $\dim D[X_1, \dots, X_m] = m + k$, $k \geq 0$. It follows from Theorem 2 of [6] that $k \geq \dim D$ so we have $D = L$ if $k = 0$. Therefore, since the theorem is true for $D = L$, we assume that $k > 0$. By Theorem 11.9 of [5, p. 37], there exists a valuation overring W of $D[X_1, \dots, X_m]$ such that $\dim W \geq m + k$. If $V = W \cap L$, then V is a valuation overring of D .

Suppose now that $\dim V = \mu$. Then by assumption $\mu \leq \omega$, and by Theorem 4 of [7], if $Z_1, Z_2, \dots, Z_r$ is any set of indeterminates over V , then $\dim V[Z_1, \dots, Z_r] = \mu + r$. In particular, if $Y_1, \dots, Y_{\mu+m}$ are indeterminates over $V[X_1, \dots, X_m]$, then $\dim V[X_1, \dots, X_m][Y_1, \dots, Y_{\mu+m}] = 2\mu + 2m$. Therefore, by Theorem 2, if $\delta_1, \dots, \delta_{\mu+m}$ are elements of $L[X_1, \dots, X_m]$, the quotient field of $V[X_1, \dots, X_m]$; then $\dim V[X_1, \dots, X_m][\delta_1, \dots, \delta_{\mu+m}] \leq \mu + m$. It then follows from Theorem 1 that every overring of $V[X_1, \dots, X_m]$ has dimension less than or equal to $\mu + m$. But W is an overring of $V[X_1, \dots, X_m]$ so we have $m + k \leq \dim W \leq m + \mu$. Therefore, $k \leq \mu \leq \omega$ so that $\dim D[X_1, \dots, X_m] \leq \omega + m$ as we wished to show.

LEMMA 3. Suppose that $\dim D[t_1, \dots, t_m] \leq \omega$ for $\{t_1, \dots, t_m\} \subseteq L$, and let P be a prime ideal of D such that $h(P) = k$. If F is the quotient field of D/P , then

$$\dim (D/P)[s_1, \dots, s_m] \leq \omega - k \quad \text{for } \{s_1, \dots, s_m\} \subseteq F.$$

Proof. F is isomorphic to D_P/PD_P , since $D_P/PD_P \cong (D/P)_{P/P}$ [8, p. 227], and D/P is isomorphic to $\{d + PD_P \mid d \in D\} \subseteq D_P/PD_P$. Thus suppose that $\{s_1, \dots, s_m\} \subseteq D_P/PD_P$ —say $s_i = t_i + PD_P$, where $t_i \in D_P$, and let $D' = D[t_1, \dots, t_m]$.

If $(0) \subset P_1 \subset \dots \subset P_k = P$ is a chain of prime ideals of D , then $(0) \subset P_1 D_P \subset \dots \subset P_k D_P = PD_P \subset D_P$ is a chain of prime ideals of D_P such that $P_i D_P \cap D = P_i$, $1 \leq i \leq k$. Now $D \subseteq D' \subseteq D_P$ so that if $P'_i = P_i D_P \cap D'$, then $P_i = P'_i \cap D$. Therefore, $(0) \subset P'_1 \subset \dots \subset P'_k = P' \subset D'$ is a chain of prime ideals of D' and $h(P') \geq k$.

It is easily seen that

$$\begin{aligned} D'/P' &\cong \{d' + PD_P \mid d' \in D'\} \\ &= \{f(t_1, \dots, t_m) + PD_P \mid f(X_1, \dots, X_m) \in D[X_1, \dots, X_m]\} \\ &= \{(d_0 + \sum d_{n_1 \dots n_m} t_1^{n_1} \dots t_m^{n_m}) + PD_P \mid d_i \in D\} \\ &\cong \{d_0 + PD_P + \sum (d_{n_1 \dots n_m} + PD_P)(t_1 + PD_P)^{n_1} \dots (t_m + PD_P)^{n_m} \mid d_i \in D\} \\ &\cong (D/P)[s_1, \dots, s_m]. \end{aligned}$$

But by assumption $\dim D' \leq \omega$, and we have seen that $h(P') \geq k$. Therefore, $\dim D'/P' \leq \omega - k$; that is, $\dim (D/P)[s_1, \dots, s_m] \leq \omega - k$, and the proof of Lemma 3 is complete.

LEMMA 4. Let P be a prime ideal of D , $P \neq D$, and let $Q_1 \subset Q_2 \subset \dots \subset Q_s$ be a chain of prime ideals of $D[X_1, \dots, X_m]$ such that $Q_i \cap D = P$ for each i , $1 \leq i \leq s$. Then $s \leq m + 1$ and there exists a chain $P[X_1, \dots, X_m] = \Gamma_1 \subset \dots \subset \Gamma_{m+1}$ of prime ideals of $D[X_1, \dots, X_m]$ such that $\Gamma_i \cap D = P$ for each i and such that $\{Q_1, \dots, Q_s\} \subseteq \{\Gamma_1, \dots, \Gamma_{m+1}\}$.

Proof. If $P^e = P[X_1, \dots, X_m]$, then $Q_1/P^e \subset \dots \subset Q_s/P^e$ is a chain of prime ideals $D[X_1, \dots, X_m]/P^e = (D/P)[X_1, \dots, X_m]$ meeting D/P in (0) . Thus, it suffices to prove Lemma 4 for the case in which $P = (0)$. But if $P = (0)$, then

$$(D[X_1, \dots, X_m])_{D-P} = L[X_1, \dots, X_m].$$

Lemma 4 now follows from the results in [9, p. 194].

LEMMA 5. *Let D be a quasi-local domain with maximal ideal M . If $\dim D = n \leq \omega \leq m$ and if $\dim D[X_1, \dots, X_m] \geq \omega + m + 1$, then there exists a chain of prime ideals of $D[X_1, \dots, X_m]$ of the form $M[X_1, \dots, X_m] \supset Q_\omega \supset \dots \supset Q_1 \supset (0)$, where either $Q_1 = P[X_1, \dots, X_m]$ for some prime ideal P of D , or $Q_1 \cap D = (0)$ but $Q_1 \cap D[X_1] \neq (0)$.*

For convenience we number the following remark since it will be used repeatedly in the proof of Lemma 5.

REMARK 1. If $\dim D[X_1, \dots, X_m] \geq \omega + m + 1$, then by Theorem 4 some overring of D has dimension greater than or equal to $\omega + 1$. Hence, by Theorem 1, we have $\dim D[t_1, \dots, t_\omega] \geq \omega + 1$ for some $\{t_1, \dots, t_\omega\} \subseteq L$ so that, by Theorem 2, $\dim D[X_1, \dots, X_\omega] \geq 2\omega + 1$.

Proof of Lemma 5. If the lemma were true in the special case in which $m = \omega$, there would exist a chain of prime ideals of $D[X_1, \dots, X_\omega]$ of the form

$$M[X_1, \dots, X_\omega] \supset Q_\omega \supset \dots \supset Q_1 \supset (0),$$

where either $Q_1 = P[X_1, \dots, X_\omega]$ for some prime ideal P of D , or $Q_1 \cap D = (0)$ but $Q_1 \cap D[X_1] \neq (0)$.

Since $\omega \leq m$,

$$M[X_1, \dots, X_m] \supset Q_\omega[X_{\omega+1}, \dots, X_m] \supset \dots \supset Q_1[X_{\omega+1}, \dots, X_m] \supset (0)$$

is a chain of prime ideals of $D[X_1, \dots, X_m]$ having the desired form.

Therefore, it suffices to prove Lemma 5 for the special case in which $\omega = m$. The proof will be by induction on n , where $n = \dim D$.

We first consider the special case in which there exists a chain of prime ideals $D[X_1, \dots, X_m] \supset Q_{2m+1} \supset \dots \supset Q_1 \supset (0)$ such that if $Q_i \cap D \neq (0)$, then $Q_i \cap D = M$, $1 \leq i \leq 2m+1$. Since this is the case when $n = 1$ we will have the first step of an induction argument.

By taking $P = (0)$ in Lemma 4, we see that $Q_{m+1} \cap D \neq (0)$ so that, by hypothesis, $Q_{m+1} \cap D = M$. Then $Q_{m+1} \supseteq M[X_1, \dots, X_m]$, and it follows from Lemma 4 that $M[X_1, \dots, X_m]$ has depth m in $D[X_1, \dots, X_m]$. But $d(Q_{m+1}) \geq m$, so $Q_{m+1} = M[X_1, \dots, X_m]$.

Since $Q_{m+1} \supset Q_m$, our assumption implies that $Q_m \cap D = (0)$. However, by Lemma 4, $Q_m \cap D[X_1] \neq (0)$ —say $Q_m \cap D[X_1] = Q'_1$. If $D' = D[X_1]_{Q'_1}$, then $D' \supseteq L$ since $Q'_1 \cap D = (0)$, and every valuation overring of D' has dimension less than or equal to one [9, p. 50]. Therefore, by Theorem 1, D' is a one-dimensional domain such that every overring has dimension less than or equal to one. It then follows

from Theorem 4 that $\dim (D')[X_2, \dots, X_m] = m$. Let $(Q'_1)^e = (Q'_1)D'$. By Lemma 4, $(Q'_1)^e[X_2, \dots, X_m]$ has depth $m-1$ in $(D')[X_2, \dots, X_m]$, so it is minimal. Since $(D')[X_2, \dots, X_m]$ is a quotient ring of $D[X_1, \dots, X_m]$ with respect to the multiplicative system

$$D[X_1] - Q'_1, Q'_1 = (Q'_1)[X_2, \dots, X_m] = (Q'_1)^e[X_2, \dots, X_m] \cap D[X_1, \dots, X_m]$$

is minimal in $D[X_1, \dots, X_m]$.

Now $Q'_1 \subseteq Q_m$ and $Q'_1 \cap D[X_1] = Q'_1 = Q_m \cap D[X_1]$. If $Q \supset Q_m$, then $h(Q) \geq m+1$, so by Lemma 4 we have $Q \cap D \neq (0)$. Hence $Q \cap D[X_1] \neq Q'_1$ since $Q'_1 \cap D = (0)$. Since Q'_1 is minimal there exists, by Lemma 4, prime ideals $Q''_2, \dots, Q''_{m-1}$ of $D[X_1, \dots, X_m]$ such that $(0) \subset Q'_1 \subset Q''_2 \subset \dots \subset Q''_{m-1} \subset Q_m$. Then $M[X_1, \dots, X_m] \supset Q_m \supset Q''_{m-1} \supset \dots \supset Q''_2 \supset Q'_1 \supset (0)$ is the desired chain.

We now assume that the result is true for $n < k$ and that $\dim D = k$. If

$$\dim D[X_1, \dots, X_m] \geq 2m+1$$

and if $D[X_1, \dots, X_m] \supset Q_{2m+1} \supset \dots \supset Q_1 \supset (0)$ is a chain of prime ideals of

$$D[X_1, \dots, X_m],$$

then, from what we have just shown, we may assume that $(0) \subset Q_i \cap D \subset M$ for some i , $1 \leq i \leq 2m+1$. Thus we choose α , $1 \leq \alpha \leq 2m+1$, such that $Q_\alpha \cap D \neq (0)$ but $Q_{\alpha-1} \cap D = (0)$. (We take $Q_0 = (0)$.) Suppose that $Q_\alpha \cap D = P$ and suppose that $\dim D_P = \mu$. By assumption $P \subset M$ so that $\mu < k$. Let $(0) \subset P_1 \subset \dots \subset P_{\mu-1} \subset P$ be a chain, having length μ , of prime ideals of D which are contained in P . Let λ be the maximal length of a proper chain of prime ideals of $D[X_1, \dots, X_m]$ which is contained properly between Q_α and $P[X_1, \dots, X_m]$ (let $\lambda = -1$ if $Q_\alpha = P[X_1, \dots, X_m]$), and let $t = (2m+1) - (\alpha-1) = 2m - \alpha + 2$. Then $P_1[X_1, \dots, X_m]$ has depth greater than or equal to $t + \lambda + \mu - 1$ in $D[X_1, \dots, X_m]$.

If $t + \lambda + \mu \geq 2m+1$, then $\dim (D/P_1)[X_1, \dots, X_m] \geq 2m = (m-1) + m+1$ and $m-1 \geq \dim D/P_1$ (since $m \geq \dim D$ and $P_1 \neq (0)$). Taking $\omega = m-1$, Remark 1 implies that $\dim (D/P_1)[X_1, \dots, X_{m-1}] \geq 2(m-1) + 1$. But $\dim D/P_1 < \dim D = k$, so by the induction hypothesis there is a chain of prime ideals of

$$(D/P_1)[X_1, \dots, X_{m-1}]$$

of the form $(M/P_1)[X_1, \dots, X_{m-1}] \supset Q''_{m-1} \supset \dots \supset Q'_1 \supset (0)$. If for $1 \leq i \leq m-1$, Q'_i is the unique prime ideal of $D[X_1, \dots, X_m]$ such that $Q'_i \supseteq P_1[X_1, \dots, X_{m-1}]$ and $(Q'_i)/P_1[X_1, \dots, X_{m-1}] \cong Q''_i$, then

$$M[X_1, \dots, X_{m-1}] \supset Q''_{m-1} \supset \dots \supset Q'_1 \supset P_1[X_1, \dots, X_{m-1}].$$

Clearly, $M[X_1, \dots, X_m] \supset (Q'_{m-1})[X_m] \supset \dots \supset (Q'_1)[X_m] \supset P_1[X_1, \dots, X_m]$ is a chain of the desired form.

We now suppose that $t + \lambda + \mu < 2m+1$. We first show the existence of a chain $M[X_1, \dots, X_m] \supset Q'_\beta \supset \dots \supset Q'_1 \supset P[X_1, \dots, X_m]$ of prime ideals of $D[X_1, \dots, X_m]$ such that $\beta + m + 1 \geq t + \lambda$.

Thus, if $t + \lambda \leq m + \dim D/P$, we take $\beta + 1 = \dim D/P$. Then there exists a chain $M \supset Q_\beta \supset \cdots \supset Q_1 \supset P$ of prime ideals of D and $M[X_1, \dots, X_m] \supset Q_\beta[X_1, \dots, X_m] \supset \cdots \supset Q_1[X_1, \dots, X_m] \supset P[X_1, \dots, X_m]$ is the desired chain. If, on the other hand, $t + \lambda \geq \dim D/P + m + 1$, we take β to be such that $t + \lambda = \beta + m + 1$. Then $\beta \geq \dim D/P$ and by assumption $t + \lambda + \mu < 2m + 1$, so that $\beta + m + 1 = t + \lambda < 2m + 1$. Therefore we have $\beta < m$; that is, $\dim D/P < m$. But $\dim (D/P)[X_1, \dots, X_m] \geq t + \lambda = \beta + m + 1$, so it follows from Remark 1 that $\dim (D/P)[X_1, \dots, X_\beta] \geq 2\beta + 1$. Since $\dim D/P < k$, the induction hypothesis is applicable. Hence, using the same method of proof given above for D/P_1 , there exist prime ideals $Q'_\beta, \dots, Q'_1$ of $D[X_1, \dots, X_m]$ such that $M[X_1, \dots, X_m] \supset Q'_\beta \supset \cdots \supset Q'_1 \supset P[X_1, \dots, X_m]$.

We now consider the domain D_P . Since $Q_i \cap D \subseteq P$ for $1 \leq i \leq \alpha$, if we set $Q_i^e = Q_i D_P[X_1, \dots, X_m]$, $1 \leq i \leq \alpha$, $P_i^e = P_i D_P$, $1 \leq i \leq \mu - 1$, and $P^e = P D_P$, then we have $(0) \subset Q_1^e \subset \cdots \subset Q_\alpha^e$, $(0) \subset (P_1^e)[X_1, \dots, X_m] \subset \cdots \subset (P_{\mu-1}^e)[X_1, \dots, X_m] \subset (P^e) \cdot [X_1, \dots, X_m] \subseteq Q_\alpha^e$, $Q_i^e \cap D_P = (0)$ for $1 \leq i \leq \alpha - 1$, $Q_\alpha^e \cap D_P = P^e$, and λ is the maximum length of a proper chain of prime ideals of $D_P[X_1, \dots, X_m]$ contained properly between Q_α^e and $(P^e)[X_1, \dots, X_m]$ ($\lambda = -1$ if $Q_\alpha^e = P^e[X_1, \dots, X_m]$). By Lemma 4 there is a chain of prime ideals of $D_P[X_1, \dots, X_m]$ of the form

$$(P^e)[X_1, \dots, X_m] = H_{m+1} \subset H_m \subset \cdots \subset H_1$$

such that $H_i \cap D_P = P^e$ for each i , and $Q_\alpha^e = H_s$ for some s , $1 \leq s \leq m + 1$. Then $(0) \subset Q_1^e \subset \cdots \subset Q_{\alpha-1}^e \subset H_s \subset \cdots \subset H_1 \subset D_P[X_1, \dots, X_m]$ is a chain of prime ideals of $D_P[X_1, \dots, X_m]$ so that $\dim D_P[X_1, \dots, X_m] \geq \alpha - 1 + s$. But by assumption $\mu + \lambda + t < 2m + 1$. Hence, $\mu + \lambda < 2m + 1 - t = \alpha - 1$ and we have $\mu + \lambda + s < \alpha - 1 + s$. By choice of the integer λ and the ideals $H_1, \dots, H_s = Q_\alpha^e$, it follows from Lemma 4 that $\lambda + s = m$. Consequently, $\mu + m < \alpha - 1 + s$. By Lemma 4, $\alpha - 1 \leq m$ (since $i \leq \alpha - 1$ implies $Q_i \cap D = (0)$), and $s \leq m + 1$ by choice. Then $\alpha - 1 + s \leq 2m + 1$, so we may choose $\gamma \leq m$ such that $\alpha - 1 + s = \gamma + m + 1$. We now have $\mu + m < \alpha - 1 + s = \gamma + m + 1 \leq \dim D_P[X_1, \dots, X_m]$, from which it follows that $\mu \leq \gamma \leq m$ (we recall that $\mu = \dim D_P$). Remark 1 now implies that $\dim D_P[X_1, \dots, X_\gamma] \geq 2\gamma + 1$.

Since $P \subset M$, $\dim D_P < k$, so by the induction hypothesis there is a chain of prime ideals of $D_P[X_1, \dots, X_\gamma]$ of the form $P^e[X_1, \dots, X_\gamma] \supset \Gamma'_\gamma \supset \cdots \supset \Gamma'_1 \supset (0)$, where either $\Gamma'_1 = P'[X_1, \dots, X_\gamma]$ for some prime ideal P' of D_P , or $\Gamma'_1 \cap D_P = (0)$ but $\Gamma'_1 \cap D_P[X_1] \neq (0)$. If we let $\Gamma_i = \Gamma'_i \cap D[X_1, \dots, X_\gamma]$, $1 \leq i \leq \gamma$, then

$$P[X_1, \dots, X_\gamma] \supset \Gamma_\gamma \supset \cdots \supset \Gamma_1 \supset (0)$$

is a chain of prime ideals of $D[X_1, \dots, X_\gamma]$. Further, if $\Gamma'_1 = (P')[X_1, \dots, X_\gamma]$, then $\Gamma_1 = (P'') [X_1, \dots, X_\gamma]$ where $P'' = P' \cap D$; or, if $\Gamma'_1 \cap D_P = (0)$ but $\Gamma'_1 \cap D[X_1] \neq (0)$, then $\Gamma_1 \cap D = (0)$ but $\Gamma_1 \cap D[X_1] \neq (0)$. We now show that

$$\begin{aligned} M[X_1, \dots, X_m] &\supset Q'_\beta \supset \cdots \supset Q'_1 \supset P[X_1, \dots, X_m] \\ &\supset L_\gamma[X_{\gamma+1}, \dots, X_m] \supset \cdots \supset \Gamma_1[X_{\gamma+1}, \dots, X_m] \supset (0) \end{aligned}$$

is the desired chain of prime ideals.

Certainly $\Gamma_1[X_{\gamma+1}, \dots, X_m]$ has the desired form, so it suffices to show that $\beta + \gamma + 1 \geq m$. But $\gamma + \lambda + s + 1 = \gamma + m + 1 = \alpha - 1 + s = 2m + 1 - t + s$. Hence, $\gamma + \lambda + 1 = 2m + 1 - t$ so that $\gamma + \lambda + t + 1 = 2m + 1$. By choice of β , $\beta + m + 1 \geq t + \lambda$ and it follows that $\gamma + \beta + m + 2 \geq \gamma + t + \lambda + 1 = 2m + 1$. Therefore, $\beta + \gamma + 1 \geq m$ as we wished to show.

This completes the proof of Lemma 5.

We are now in position to show that the conditions given in Theorem 2 are necessary in order that $\dim D[t_1, \dots, t_m] \leq \omega$ for $\{t_1, \dots, t_m\} \subseteq L$. Theorem 5 is the main result of this paper.

THEOREM 5. *If $\dim D = n$ and if m, ω are nonnegative integers such that*

$$\dim D[t_1, \dots, t_m] \leq \omega \quad \text{for } \{t_1, \dots, t_m\} \subseteq L,$$

then the following conditions hold.

- (1) $\dim D[X_1, \dots, X_m] \leq \omega + m$.
- (2) *If there exist elements $t_1, \dots, t_m$ in L such that $\dim D[t_1, \dots, t_m] = \omega$, then $\dim D[X_1, \dots, X_m] = \omega + m$.*

Proof of (1). The proof of (1) will be by induction on n and m . Thus, we first show that (1) is true when either $n = 1$ or $m = 1$.

Suppose that $n = 1$. By a theorem of Sedenberg [7, p. 608], D one-dimensional implies that for any m , $\dim D[X_1, \dots, X_m] \leq 2m + 1$. Clearly then (1) holds if $\omega \geq m + 1$, so we assume that $\omega \leq m$. Since $\dim D[t_1, \dots, t_m] \leq \omega$ for $\{t_1, \dots, t_m\} \subseteq L$, it follows, by taking $t_\omega = t_{\omega+1} = \dots = t_m$, that $\dim D[t_1, \dots, t_\omega] \leq \omega$ for $\{t_1, \dots, t_\omega\} \subseteq L$. Theorem 1 now implies that each overring of D has dimension less than or equal to ω so that, by Theorem 4 $\dim D[X_1, \dots, X_m] \leq m + \omega$.

Now suppose that $m = 1$. We have just seen that (1) holds for $n = 1$, so we assume that (1) is true for $n < h$, that $\dim D = h$, and that $\dim D[t] \leq \omega$ for $t \in L$. Let $(0) \subset Q_1 \subset \dots \subset Q_s \subset D[X]$ be a chain of prime ideals of $D[X]$, where Q_1 is chosen to be minimal. If $Q_1 \cap D = (0)$, then $D[X]/Q_1 \cong D[\bar{X}]$, where $\bar{X} = X + Q_1$, and $f(\bar{X}) = 0$ for any $f \in Q_1$, so $\bar{X}$ is algebraic over D . It follows from Corollary 1 that $\dim D[\bar{X}] \leq \omega$ and this implies that Q_1 has depth less than or equal to ω in $D[X]$. Therefore, $s \leq \omega + 1$. On the other hand, if $Q_1 \cap D \neq (0)$, then $Q_1 \cap D \supseteq P$, where P is a minimal prime ideal of D . By choice, Q_1 is minimal, and $Q_1 \supseteq P[X]$. Thus, $Q_1 = P[X]$. From Lemma 3 we have $\dim (D/P)[\sigma] \leq \omega - 1$ for each σ in the quotient field of D/P . Further, $\dim (D/P) < h$, and by assumption (1) holds; that is,

$$\dim (D/P)[X] \leq \omega.$$

But $(D/P)[X] \cong D[X]/P[X]$ so that $P[X] = Q_1$ has depth less than or equal to ω . Consequently $s \leq \omega + 1$ and it follows by induction that (1) is true for $m = 1$.

From what we have just shown, we may make the following inductive assumptions.

- (A) Suppose that (1) is true for any n when $m < k$.

(B) Suppose that (1) is true for $n < h$ when $m = k$.

Now let $\dim D = h$ and suppose that $\dim D[X_1, \dots, X_k] \geq \omega + k + 1$, $\omega \geq h$. We wish to establish the existence of $t_1, \dots, t_k$ in L such that $\dim D[t_1, \dots, t_k] \geq \omega + 1$.

If $\dim D[X] \geq \omega + 2$, then by the case in which $m = 1$ there exists $t \in L$ such that $\dim D[t] \geq \omega + 1$. If we set $t = t_1 = \dots = t_k$, it follows that $\dim D[t_1, \dots, t_k] \geq \omega + 1$ and we are finished.

Suppose, then, that $\dim D[X_1] \leq \omega + 1$. The assumption that

$$\dim D[X_1][X_2, \dots, X_k] \geq \omega + k + 1$$

then implies, by (A), that there exist elements $\delta_2, \dots, \delta_k$ in $L(X_1)$ such that $\dim D[X_1][\delta_2, \dots, \delta_k] \geq \omega + 2$. Let Q be the kernel of the canonical $D[X_1]$ -homomorphism ϕ which maps $D[X_1][X_2, \dots, X_k]$ onto $D[X_1][\delta_2, \dots, \delta_k]$ in such a way that $\phi(X_i) = \delta_i$ for each i . Then Q must have depth greater than or equal to $\omega + 2$, and by Lemma 1, Q has height $k - 1$. Hence, there exists a chain of prime ideals of $D[X_1, \dots, X_k]$ of the form

$$(0) \subset Q_1 \subset \dots \subset Q_{k-2} \subset Q \subset Q_k \subset \dots \subset Q_{k+\omega+1} \subset D[X_1, \dots, X_k].$$

If $f(X_1) \in D[X_1]$, then $\phi(f(X_1)) = f(X_1)$. Therefore, $Q \cap D[X_1] = (0)$. However, since $h(Q) = k - 1$, Lemma 4 implies that $Q_k \cap D[X_i] \neq (0)$ for i , $1 \leq i \leq k$. We now consider the two cases in which $Q_k \cap D = (0)$ and $Q_k \cap D \neq (0)$.

If $Q_k \cap D = (0)$, then $D[X_1, \dots, X_k]/Q_k \cong D[\bar{X}_1, \dots, \bar{X}_k]$, where $\bar{X}_i = X_i + Q_k$, and since $Q_k \cap D[X_i] \neq (0)$, $\bar{X}_i$ is algebraic over D for each i . But

$$\dim D[\bar{X}_1, \dots, \bar{X}_k] \geq \omega + 1,$$

so by Corollary 1 there exist elements $t_1, \dots, t_k$ in L such that $\dim D[t_1, \dots, t_k] \geq \omega + 1$ and we are finished.

Thus, suppose that $Q_k \cap D \neq (0)$ —say $Q_k \cap D = P$, where P is a prime ideal of D such that $h(P) = \mu$. Then $Q_k \supseteq P[X_1, \dots, X_k]$ and there exists a chain $(0) \subset P_1 \subset \dots \subset P_{\mu-1} \subset P$ of prime ideals of D . Let λ be the maximal length of a proper chain of prime ideals of $D[X_1, \dots, X_k]$ contained properly between Q_k and $P[X_1, \dots, X_k]$ (let $\lambda = -1$ if $Q_k = P[X_1, \dots, X_k]$). We proceed now to show the existence of a chain of prime ideals of $D[X_1, \dots, X_k]$ which has length greater than or equal to $k + \omega + 1$ and which is of the form $Q_{k+\omega+1} \supset \dots \supset Q_k \supseteq \dots \supseteq P[X_1, \dots, X_k] \supset Q'_\gamma \supset \dots \supset Q'_1 \supset (0)$ where either $Q'_1 = P'[X_1, \dots, X_k]$ for some prime ideal P' of D , or $Q'_1 \cap D = (0)$ but $Q'_1 \cap D[X_1] \neq (0)$. (Here we understand that a proper chain of length λ is contained between Q_k and $P[X_1, \dots, X_k]$.) We shall say that such a chain has form (C). Thus, we seek a chain of form (C) for which $\gamma + \lambda + \omega + 3 \geq k + \omega + 1$.

If $\mu + \lambda \geq k - 1$, then

$$\begin{aligned} Q_{k+\omega+1} &\supset \dots \supset Q_k \supseteq \dots \supseteq P[X_1, \dots, X_k] \supset P_{\mu-1}[X_1, \dots, X_k] \\ &\supset \dots \supset P_1[X_1, \dots, X_k] \supset (0) \end{aligned}$$

(where a proper chain of length λ is included between Q_k and $P[X_1, \dots, X_k]$) is such a chain since $(\mu - 1) + \lambda + \omega + 3 = \mu + \lambda + \omega + 2 \geq k - 1 + \omega + 2 = k + \omega + 1$.

Suppose, then, that $\mu + \lambda < k - 1$ and consider the domain D_P . For $1 \leq i \leq k$, let $Q_i^e = Q_i D_P[X_1, \dots, X_k]$, and for $1 \leq i \leq \mu$, let $P_i^e = P_i D_P$, where we set $Q_{k-1} = Q$ and $P_\mu = P$. Then $(0) \subset Q_1^e \subset \dots \subset Q_k^e$ is a chain of prime ideals of D_P , $Q_i^e \cap D_P = (0)$ for $1 \leq i \leq k - 1$, $Q_k^e \supseteq (P^e)[X_1, \dots, X_k]$, $Q_k^e \cap D_P = P^e$, and λ is the maximal length of a chain of prime ideals of $D_P[X_1, \dots, X_k]$ contained properly between Q_k^e and $P^e[X_1, \dots, X_k]$. By Lemma 4 there is a chain of prime ideals $P^e[X_1, \dots, X_k] = \Gamma_{k+1} \subset \Gamma_k \subset \dots \subset \Gamma_1 \subset D_P[X_1, \dots, X_k]$ such that $\Gamma_i \cap D_P = P^e$ for each i and such that $Q_k^e = \Gamma_s$ for some s , $1 \leq s \leq k + 1$. From Lemma 4, and by choice of λ , it follows that $\lambda = k - s$; that is, $s + \lambda = k$. We now have the chain $(0) \subset Q_1 \subset \dots \subset Q_{k-1} \subset \Gamma_s \subset \dots \subset \Gamma_1 \subset D_P[X_1, \dots, X_k]$ of prime ideals of $D_P[X_1, \dots, X_k]$, from which it follows that $\dim D_P[X_1, \dots, X_k] \geq s + k - 1 > s + \lambda + \mu = k + \mu$. Let γ be chosen so that $s + k - 1 = k + \gamma + 1$. Then $\gamma \geq \mu = \dim D_P$ and, by choice of s , we have $s \leq k + 1$, so that $k + \gamma + 1 = s + k - 1 \leq 2k$. Consequently, we have $\mu \leq \gamma \leq k - 1$ and $\dim D_P[X_1, \dots, X_k] \geq k + \gamma + 1$. Therefore, by Lemma 5, there exists a chain of prime ideals of $D_P[X_1, \dots, X_k]$ of the form $P^e[X_1, \dots, X_k] \supset Q'_\gamma \supset \dots \supset Q'_1 \supset (0)$, where either $Q'_1 = P''[X_1, \dots, X_k]$ for some prime ideal P'' of D_P , or $Q'_1 \cap D_P = (0)$ but $Q'_1 \cap D_P[X_1] \neq (0)$. Let $Q'_i = Q'_i \cap D[X_1, \dots, X_k]$ for each i , $1 \leq i \leq \gamma$. Then $Q_{\omega+k+1} \supset \dots \supset Q_k \supset \dots \supset P[X_1, \dots, X_k] \supset Q'_\gamma \supset \dots \supset Q'_1 \supset (0)$ is a chain of prime ideals of $D[X_1, \dots, X_k]$ having form (C), for if $Q'_1 = P''[X_1, \dots, X_k]$ for some prime ideal P'' of D_P , then $Q'_1 = P'[X_1, \dots, X_k]$, where $P' = P'' \cap D$. On the other hand, if $Q'_1 \cap D_P = (0)$ but $Q'_1 \cap D_P[X_1] \neq (0)$, then $Q'_1 \cap D = (0)$ but $Q'_1 \cap D[X_1] \neq (0)$. Further, $s + k - 1 = k + \gamma + 1 = \lambda + s + \gamma + 1$ so that $k - 1 = \lambda + \gamma + 1$. It then follows that $k + \omega + 1 = \lambda + \gamma + \omega + 3$.

LEMMA 6. Suppose that $\dim D = h$ and $\dim D[t_1, \dots, t_k] \leq \omega$ for $\{t_1, \dots, t_k\} \subseteq L$. Then if P is a proper prime ideal of D , $P[X_1, \dots, X_k]$ has depth less than or equal to $\omega + k - 1$ in $D[X_1, \dots, X_k]$, and if Q is a prime ideal of $D[X_1, \dots, X_k]$ such that $Q \cap D = (0)$ but $Q \cap D[X_1] \neq (0)$, then Q has depth less than or equal to $\omega + k - 1$ in $D[X_1, \dots, X_k]$.

Proof. If P is a proper prime ideal of D , then by Lemma 3 we have

$$\dim (D/P)[s_1, \dots, s_k] \leq \omega - h(P)$$

for any set of elements $\{s_1, \dots, s_k\}$ contained in the quotient field of D/P . From assumption (B) it then follows that $\dim (D/P)[X_1, \dots, X_k] \leq \omega + k - h(P)$. But $(D/P)[X_1, \dots, X_k] \cong D[X_1, \dots, X_k]/P[X_1, \dots, X_k]$, so that $P[X_1, \dots, X_k]$ has depth less than or equal to $\omega + k - h(P)$.

Suppose that Q is a prime ideal of $D[X_1, \dots, X_k]$ such that $Q \cap D = (0)$ but $Q \cap D[X_1] \neq (0)$ —say $Q \cap D[X_1] = Q'$. Then $Q \supseteq (Q')[X_2, \dots, X_k]$ and

$$D[X_1, \dots, X_k]/(Q')[X_2, \dots, X_k] \cong (D[X_1]/Q')[X_2, \dots, X_k].$$

But $D[X_1]/Q' \cong D[\bar{X}_1]$, where $\bar{X}_1 = X_1 + Q'$, and $\bar{X}_1$ is algebraic over D . Since $\dim D[t] \leq \omega$ for $t \in L$, it follows from Corollary 1 that $\dim D[\bar{X}_1] \leq \omega$. Moreover, by Lemma 2, there exists a nonzero element d in D such that $\bar{X}_1$ is integral over $D[1/d]$. But L is the quotient field of $D[1/d]$ and $\dim D[1/d][t_1, \dots, t_{k-1}] \leq \omega$ for $\{t_1, \dots, t_{k-1}\} \subseteq L$. Therefore, by Theorem 3, we have $\dim D[\bar{X}_1][s_1, \dots, s_{k-1}] \leq \omega$ for any set of elements $\{s_1, \dots, s_{k-1}\}$ of the quotient field of $D[\bar{X}_1]$. It now follows from assumption (A) that $\dim D[\bar{X}_1][X_2, \dots, X_k] \leq \omega + k - 1$. Consequently, $Q'[X_2, \dots, X_k]$ must have depth less than or equal to $\omega + k - 1$ so that Q also has depth less than or equal to $\omega + k - 1$ as we wished to show.

We now complete the proof of Theorem 5.

By assumption the ideal Q'_1 in a chain having form (C) has depth greater than or equal to $k + \omega$. However, Q'_1 has one of the forms described in Lemma 6 so it follows that $\dim D[t_1, \dots, t_k] \geq \omega + 1$ for some set $\{t_1, \dots, t_k\} \subseteq L$.

Statement (1) of Theorem 5 now follows by induction.

Assume now that $\dim D[t_1, \dots, t_m] = \omega$ for some $\{t_1, \dots, t_m\} \subseteq L$. From (1) it follows that $\dim D[X_1, \dots, X_m] \leq \omega + m$. But if $\dim D[X_1, \dots, X_m] = \alpha + m$, where $\alpha \leq \omega$, it follows from Theorem 2 that $\dim D[s_1, \dots, s_m] \leq \alpha$ for $\{s_1, \dots, s_m\} \subseteq L$. In particular, $\dim D[t_1, \dots, t_m] = \omega \leq \alpha$, so that $\alpha = \omega$. Statement (2) of Theorem 5 now follows.

This completes the proof of Theorem 5.

In [3] Jaffard defines the *valuative dimension*, denoted by $\dim_v D$, of the domain D to be the maximal rank of the valuation overrings of D . With this notation and terminology, we now relate many of the results of this paper in the following theorem.

THEOREM 6. *Let D be a finite-dimensional integral domain with identity having quotient field L , and let ω be a positive integer such that $\omega \geq \dim D$. Then the following statements are equivalent.*

- (1) $\dim_v D = \omega$.
- (2) *Each overring of D has dimension less than or equal to ω and ω is minimal.*
- (3) *For any nonnegative integer m , $\dim D[t_1, \dots, t_m] \leq \omega$ for $\{t_1, \dots, t_m\} \subseteq L$, and for $m \geq \omega - 1$ there exists $\{t_1, \dots, t_m\} \subseteq L$ such that $\dim D[t_1, \dots, t_m] = \omega$.*
- (4) *For any nonnegative integer m , $\dim D[X_1, \dots, X_m] \leq m + \omega$ and for $m \geq \omega - 1$ equality holds.*
- (5) $\dim D[X_1, \dots, X_\omega] = 2\omega$.
- (6) $\dim D[t_1, \dots, t_\omega] \leq \omega$ for any set $\{t_1, \dots, t_\omega\} \subseteq L$, and there exists a set $\{s_1, \dots, s_\omega\} \subseteq L$ such that $\dim D[s_1, \dots, s_\omega] = \omega$.

Proof. It was shown in the proof of Theorem 1 that if D' is an overring of D such that $\dim D' = k$, then there exists a valuation overring V of D such that $\dim V \geq k$. This fact together with Theorem 1 shows that (1) and (2) are equivalent.

To show that (2) implies (3), it clearly suffices to show that for any positive integer $m \geq \omega - 1$, there exists $\{t_1, \dots, t_m\} \subseteq L$ such that $\dim D[t_1, \dots, t_m] = \omega$.

However, it follows from the proof of Theorem 1 that if there exists an overring D' of D such that $\dim D' = \omega$, then $\dim D[t_1, \dots, t_{\omega-1}] \geq \omega$ for some $\{t_1, \dots, t_{\omega-1}\} \subseteq L$. Thus, equality holds and for any $m \geq \omega - 1$, $\dim D[t_1, \dots, t_m] = \omega$, where $t_{\omega-1} = t_\omega = \dots = t_m$.

That (3) implies (4) is an immediate consequence of Theorem 5 and certainly (4) implies (5). If (5) holds, then by Theorem 2 we have $\dim D[t_1, \dots, t_\omega] \leq \omega$ for $\{t_1, \dots, t_\omega\} \subseteq L$. But if $\dim D[t_1, \dots, t_\omega] \leq k$ for any $k \leq \omega$, then it follows from Theorem 5 that $\dim D[X_1, \dots, X_\omega] \leq k + \omega$ so that $k \geq \omega$. Thus $k = \omega$, and it follows that $\dim D[s_1, \dots, s_\omega] = \omega$ for some $\{s_1, \dots, s_\omega\} \subseteq L$. Therefore (6) holds.

It is immediate from Theorem 1 that (6) implies (2) and Theorem 6 is proved.

REMARK 2. If we take $\omega = \dim D$, then for any nonnegative integer m and $\{t_1, \dots, t_m\} \subseteq L$, we have $\dim D[t_1, \dots, t_m] = \omega$. Thus from Theorem 6,

$$\dim D[X_1, \dots, X_m] = m + \dim D$$

for all m if and only if $\dim D = \dim_v D$.

II. Suppose now that D is integrally closed. Let $\{V_\alpha\}$ be the set of all valuation overrings of D , and let A be an ideal of D . Then $\tilde{A} = \bigcap_\alpha AV_\alpha$ is an ideal of D called the *completion* of A . If X is an indeterminate over D and $f \in D[X]$, then we denote by A_f the ideal of D generated by the coefficients of f . We now define the Kronecker function ring of D as follows:

$$D^K = \{f/g \mid f, g \in D[X], \tilde{A}_f \subseteq \tilde{A}_g\}.$$

In [4], Krull shows that D^K is an integral domain having quotient field $L(X)$ and that $D^K \cap L = D$. He further showed that D^K is a Bezout domain, where a *Bezout domain* is defined to be a domain in which each finitely generated ideal is principal.

Now let V be a valuation overring of D and let v be a valuation associated with V . If $f \in L[X] - \{0\}$, $f = f_0 + f_1X + \dots + f_nX^n$, we define $v^*(f) = \min_{0 \leq i \leq n} \{v(f_i) \mid f_i \neq 0\}$. Then v^* defines a valuation on $L(X)$ having the same value group as v . In particular, v and v^* have the same rank. We call v^* the *trivial extension of v to $L(X)$* , and if V^* is the valuation ring of $L(X)$ associated with v^* , then V^* is called the *trivial extension of V to $L(X)$* . Krull has shown in [4, p. 560] that if $\{V_\alpha\}$ is the collection of valuation overrings of D , then $\{V_\alpha^*\}$ is the collection of valuation overrings of D^K .

An integral domain R with identity is said to be a *Prüfer domain* provided each finitely generated nonzero ideal of R is invertible. In particular, a Bezout domain is a Prüfer domain, so D^K is a Prüfer domain. Therefore, $\dim_v D^K = \dim D^K$ [3, p. 56]. But from the previous remarks we see that $\dim_v D = \dim_v D^K$. We have thus proved the following result.

THEOREM 7. *Let D be an integrally closed domain with identity and let D^K be the Kronecker function ring of D . Then $\dim_v D = \dim D^K$.*

COROLLARY 2. *If D is an integral domain with identity having integral closure $\bar{D}$, the statement that $\dim(\bar{D})^K = \omega$ is equivalent to each of the statements (1)–(6) of Theorem 6.*

III. Let D be an n -dimensional integral domain with identity having quotient field L . We have seen that each overring of D has dimension less than or equal to n if and only if $\dim D[t_1, \dots, t_n] \leq n$ for each subset $\{t_1, \dots, t_n\} \subseteq L$. For any positive integer n , we now show the existence of an integral domain D such that $\dim D = n$, and such that $\dim D[t_1, \dots, t_m] \leq n$ for any positive integer $m < n$ and for each subset $\{t_1, \dots, t_m\} \subseteq L$, but such that $\dim V = n + 1$ for some valuation overring V of $D^{(2)}$. We first state the following results which are proved in [2].

LEMMA 7. Let $\{V_1, \dots, V_k\}$ be a collection of valuation rings having quotient field L , and suppose that $V_i \not\subseteq V_j$ for $i \neq j$. If M_i is the maximal ideal of V_i , then $\bigcap_{j \neq i} M_j \not\subseteq V_i$ for any i .

LEMMA 8. Let $\{V_1, \dots, V_k\}$ be as in Lemma 7 and suppose that each V_i contains some fixed field F . If $D = F + M$, where $M = M_1 \cap \dots \cap M_k$, then D is a quasi-local domain with maximal ideal M and if P is a nonmaximal prime ideal of D , then $P = Q \cap D$, where Q is a nonmaximal prime ideal of V_i for some i , $1 \leq i \leq k$.

Now let n be an arbitrary positive integer, let K be a field, and let $L = K(X_1, \dots, X_{n+1})$. We may construct valuation rings V_1 and V_2 on L such that:

(a) V_1 has rank one and $V_1 = K(X_1, \dots, X_n) + M_1$, where M_1 is the maximal ideal of V_1 , and $X_{n+1} \in M_1$.

(b) V_2 has rank n , $V_2 = K + M_2$, where M_2 is the maximal ideal of V_2 , $X_1/X_{n+1} \in M_2$, and if $M_2 = P_1 \supset P_2 \supset \dots \supset P_n \supset P_{n+1} = (0)$ is the chain of prime ideals of V_2 , then $X_i \in P_i - P_{i+1}$ for each i , $1 \leq i \leq n$.

We have $X_1/X_{n+1} \in V_2 - V_1$ and $1/X_1 \in V_1 - V_2$. Thus, by Lemma 8, if $D = K + M$, where $M = M_1 \cap M_2$, then D is a quasi-local domain with maximal ideal M , and D has quotient field L since M does. Further, $X_i X_{n+1} \in (P_i \cap D) - (P_{i+1} \cap D)$ for each i , $1 \leq i \leq n$, so it follows from Lemma 8 that $\dim D = n$.

Suppose that V is a nontrivial valuation overring of D . Then $V \supseteq M_1 \cap M_2$, so by Lemma 7 either $V \subseteq V_i$ or $V \supseteq V_i$ for $i = 1$ or 2 . If $V \supseteq V_1$, then $V = V_1$ since $\dim V_1 = 1$. If $V \subseteq V_2$, then $V \supseteq M_2$, and $V \supseteq K$ since $V \supseteq D$. Therefore, $V \supseteq K + M_2 = V_2$, so that equality holds. Thus, if V is a nontrivial valuation overring of D , either $V \subseteq V_1$ or $V \supseteq V_2$.

Let m be a positive integer, $m < n$, and let $\{t_1, \dots, t_m\} \subseteq L$. Then $D[t_1, \dots, t_m]$ is the homomorphic image of $D[Y_1, \dots, Y_m]$, $Y_1, \dots, Y_m$ indeterminates over D , so it follows from Lemma 4 that if $P_1 \subset P_2 \subset \dots \subset P_s$ is a chain of prime ideals of $D[t_1, \dots, t_m]$ such that $P_i \cap D = M$ for each i , then $s \leq m + 1 \leq n$. Further, let D' be an overring of D such that $\dim D' \geq n + 1$, and let $(0) \subset P'_1 \subset \dots \subset P'_{n+1} \subset D'$ be a chain of prime ideals of D' . Then there exists a valuation overring V of D' , and a chain $(0) \subset Q_1 \subset \dots \subset Q_{n+1} \subset V$ of prime ideals of V such that $Q_i \cap D' = P'_i$, $1 \leq i \leq n + 1$ [5, p. 37]. Since $\dim V \geq n + 1$, $V \not\subseteq V_2$. Therefore, $V \subseteq V_1$, so it follows

(²) The method for constructing such an example was suggested by William Heinzer.

that $M_1 \subseteq Q_1$. Thus $P'_1 \cap D = (Q_1 \cap D') \cap D \supseteq (M_1 \cap D') \cap D = M$, and consequently, $P'_i \cap D = M$ for each i , $1 \leq i \leq n+1$. From what we have shown it follows that $\dim D[t_1, \dots, t_m] \leq n$ for $\{t_1, \dots, t_m\} \subseteq L$. But we may construct a valuation ring V_3 on L such that V_3 has rank $n+1$ and $V_3 = K + M_3$, where M_3 is the maximal ideal of V_3 and $M_3 \supseteq M_1$. Then $V_3 \supseteq K + M_1 \supseteq D$.

REFERENCES

1. R. W. Gilmer, Jr., *Domains in which valuation ideals are prime powers*, Arch. Math. **17** (1966), 210–215.
2. R. W. Gilmer, Jr. and William Heinzer, *Primary ideals and valuation ideals. II*, Trans. Amer. Math. Soc. **131** (1968), 149–162.
3. P. Jaffard, *Théorie de la dimension dans les anneaux de polynomes*, Gauthier-Villars, Paris, 1960.
4. W. Krull, *Beitrage zur Arithmetik kommutativer Integritätsbereiche*, Math. Z. **41** (1936), 545–577.
5. M. Nagata, *Local rings*, Interscience, New York, 1962.
6. A. Seidenberg, *A note on the dimension theory of rings*, Pacific J. Math. **3** (1953), 505–512.
7. ———, *A note on the dimension theory of rings. II*, Pacific J. Math. **4** (1954), 603–614.
8. O. Zariski and P. Samuel, *Commutative algebra*, Vol. 1, Van Nostrand, Princeton, N. J., 1958.
9. ———, *Commutative algebra*, Vol. 2, Van Nostrand, Princeton, N. J., 1960.

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